Quantum noise

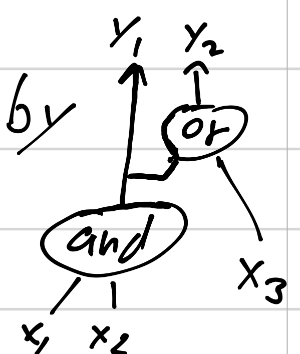
Recap: Deutsch alg: $f:\{0,1\} \to \{0,1\}$ $\quad f(0) \overset{?}{=} f(1)$

$$\sum_{x\in\{0,1\}} \tfrac{1}{\sqrt{2}}|x,0\rangle \longrightarrow \sum_{x\in\{0,1\}} \tfrac{1}{\sqrt{2}}|x, f(x)\rangle = \tfrac{1}{\sqrt{2}}\big(|0,f(0)\rangle + |1,f(1)\rangle\big)$$

$b = f(0) \in \{0,1\}$

can dist. $\frac{1}{\sqrt{2}}\big(|0,b\rangle + |1,b\rangle\big)$ from $\frac{1}{\sqrt{2}}\big(|0,b\rangle + |1,\bar{b}\rangle\big)$

more generally: let $f:\{0,1\}^n \to \{0,1\}^n$ computed by a boolean circuit with t gates.

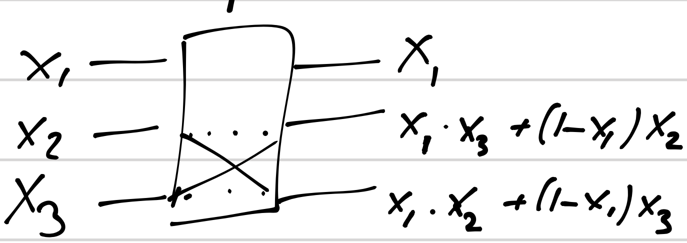


Thm: $\sqrt{\frac{1}{2^n}}\sum_{x\in\{0,1\}^n} |x,0\rangle \xrightarrow{\ 2t+n \text{ steps}\ } \sqrt{\frac{1}{2^n}}\sum_{x\in\{0,1\}^n} |x, f(x)\rangle$

How? step 1: make computation of f reversible.

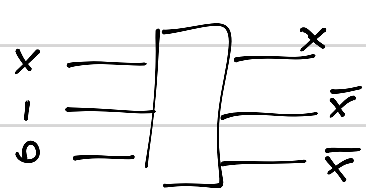
$x_1, x_2, x_3 \in \{0,1\}$

Fredkin gate:

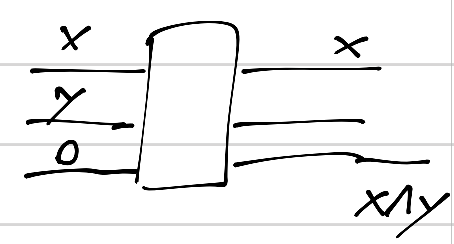


clearly reversible & unitary on $\mathbb{C}^8$.

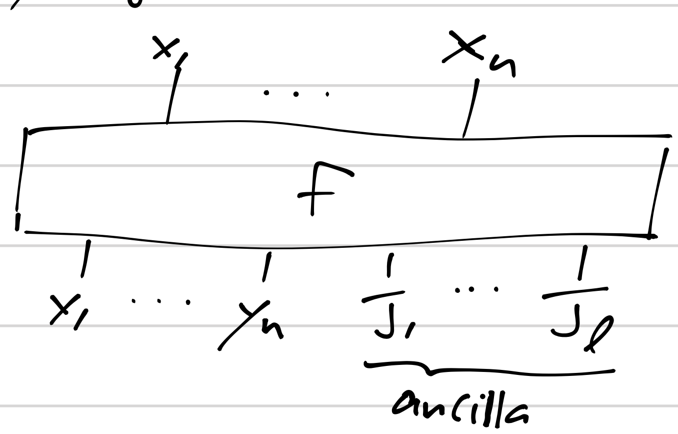
NOT(x):



AND(x,y):



Replace every gate in circuit for f by Fredkin gate:



$\Leftarrow$ reversible

Step 2: $\sqrt{\frac{1}{2^n}}\sum_{x\in\{0,1\}^n} |x, 0^\ell, 0^n\rangle \xrightarrow{t} \sqrt{\frac{1}{2^n}}\sum_{x\in\{0,1\}^n} |f(x), J, 0^n\rangle \longrightarrow$

n CNOT gates $\longrightarrow \sqrt{\frac{1}{2^n}}\sum_{x\in\{0,1\}^n} |f(x), J, f(x)\rangle \xrightarrow[\text{reverse}]{t} \sqrt{\frac{1}{2^n}}\sum_{x\in\{0,1\}^n} |x, 0^\ell, f(x)\rangle$

Quantum noise

Recall: Deutsch's alg returns $\begin{cases} Pr[0]=1 & \text{if } f(0)=f(1) \\ Pr[1]=1 & \text{if } f(0)\neq f(1) \end{cases}$

but when running 1000 times on Q.C. only got 90% 1. why?

Two types of noise:

coherent noise: $|\psi\rangle \to U\cdot|\psi\rangle$ but instead $|\psi\rangle \to \tilde{U}|\psi\rangle$
($\tilde{U}$ close to U)

$\Rightarrow$ incoherent noise: due to interaction with env.

Density matrix: suppose we only know that

$$Pr[\text{state} = \psi_1] = \tfrac{1}{3}, \quad Pr[\text{state} = \psi_2] = \tfrac{2}{3}$$

we write system as enamble $\{(\frac{1}{3}, \psi_1), (\frac{2}{3}, \psi_2)\}$

more generally $\{(p_i, \psi_i)\}_{i=1}^{n}$ $\quad p_1 + \dots + p_n = 1$ positive.

$n=1$: pure state, $\quad n>1$: mixed state.

ex: $\underbrace{\{(\tfrac{1}{2}, |0\rangle), (\tfrac{1}{2}, |1\rangle)\}}_{\text{mixed state}}$ $(*)$ $\quad \Leftarrow$ one of two classical states (coin flip)

Def: density matrix $\rho := \sum_{i=1}^{n} p_i \cdot |\psi_i\rangle \cdot \langle \psi_i| \in \mathbb{C}^{d\times d}$
$d = \dim(\psi_i)$

examples: (1) $\psi = \psi_0|0\rangle + \psi_1|1\rangle$ (pure) $\Rightarrow \rho = \begin{pmatrix} |\psi_0|^2 & \psi_0\psi_1^* \\ \psi_0^*\psi_1 & |\psi_1|^2 \end{pmatrix}$

(2) $(*)$ $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{I}{2}$

Properties of ρ: (1) $tr(\rho) = 1$, (2) $\rho^2 = \rho \Leftrightarrow \rho$ is pure

(3) different encambles can give same ρ (some info. loss)

Why use ρ? (1) if ρ, σ are density matrices of indep. systems

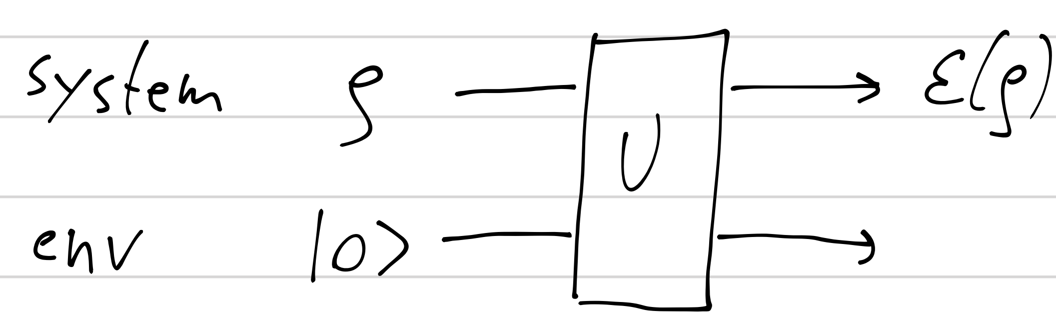
then $\rho \otimes \sigma \in \mathbb{C}^{d_\rho d_\sigma \times d_\rho d_\sigma}$ is density matrix of composition

(2) For a unitary state transition U on $\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$

$\Longrightarrow$ resulting mixed state is $\boxed{U\rho U^\dagger}$ why?

$$\sum_{i=1}^{n} p_i |U\psi_i\rangle\langle U\psi_i| = \sum_{i=1}^{n} p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\left[\sum_{i=1}^{n} p_i|\psi_i\rangle\langle\psi_i|\right]U^\dagger = U\rho U^\dagger$$

Incoherent noise: want to act on system, but can only act on system + env.



env is initially indep. of system. Can assume is $|0\rangle$.

main point: $U : H_S \otimes H_E \to H_S \otimes H_E$ operates on system + env.

$\mathcal{E} : H_S \to H_S$ operates on H_S

Thm: $\mathcal{E}(\rho) = \sum_{i=1}^{k} E_i \cdot \rho \cdot E_i^\dagger$, $\sum_{i=1}^{k} E_i^\dagger E_i = I$

E_i: Kraus operators, operate on H_S.

Noise models:

(1) Bit flip:
(prob. $1-p$)

$$\overbrace{\psi := \psi_0|0\rangle + \psi_1|1\rangle}^{\text{pure state}} \longrightarrow \{(p, \psi), (1-p, X\psi)\}$$

mixed state

$X\psi = \psi_1|0\rangle + \psi_0|1\rangle$

As density matrix:

$$\rho = |\psi\rangle\langle\psi| \implies \mathcal{E}(\rho) = p|\psi\rangle\langle\psi| + (1-p)|X\psi\rangle\langle X\psi| =$$
$$= p\rho + (1-p)X\rho X$$

$$\implies E_0 = \sqrt{p}\, I, \quad E_1 = \sqrt{1-p}\, X$$

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

(2) Phase flip:
(prob. $1-p$)

$$|\psi\rangle \to \{(p, |\psi\rangle), (1-p, Z|\psi\rangle)\}$$

$Z|\psi\rangle = \psi_0|0\rangle - \psi_1|1\rangle$

$$E_0 = \sqrt{p}\, I, \quad E_1 = \sqrt{1-p}\, Z$$

(3) Depolarizing noise:
bit is randomized w/prob. p.

$$|\psi\rangle \to \left\{\left(\tfrac{p}{2}, |0\rangle\right), \left(\tfrac{p}{2}, |1\rangle\right), \left((1-p), |\psi\rangle\right)\right\}$$

$$\mathcal{E}(\rho) = \underbrace{\frac{pI}{2}}_{\text{depolarize w/prob. } p} + (1-p)\rho = \underbrace{\frac{p}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z) + (1-p)\rho}_{\text{5 Kraus operators}}$$

(4) Amplitude dampening: energy loss (radiate photon) w/prob. γ

applied to classical states:

$|0\rangle \to |0\rangle, \quad |1\rangle \to \sqrt{\gamma}\,|0\rangle + \sqrt{1-\gamma}\,|1\rangle$

Kraus operators: $\quad E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$

$\rho \to E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$

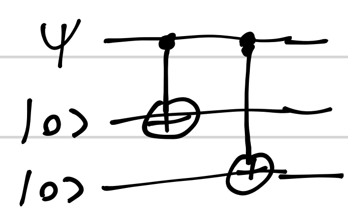
Quantum error correction

Goal: quantum computing despite errors.

Step 1: maintain state in presence of single bit errors.

qbit $\psi = \psi_0|0\rangle + \psi_1|1\rangle$. 3-bit code: $\boxed{\hat{\psi} = \psi_0|000\rangle + \psi_1|111\rangle}$

encoding:



bit flip error on 1^{st} bit:

$\psi_0|100\rangle + \psi_1|011\rangle$

How to decode?

observable Z_1Z_2: eigenspaces $\begin{cases} +: \text{span}(|00*\rangle, |11*\rangle) & \leftarrow \dim = 4 \\ -: \text{span}(|01*\rangle, |10*\rangle) \end{cases}$

measuring Z_1Z_2: $+ \Rightarrow$ first two bits equal
$- \Rightarrow$ — " — — " — unequal

measuring Z_2Z_3: $+ \Rightarrow$ bits 2 & 3 equal
$- \Rightarrow$ — " — unequal

} does not change state $\hat{\psi}$! does not reveal bit values.

$++$: no error, $\begin{cases} -+: 1^{st} \text{ bit flipped, apply } X_1 \text{ to correct} \\ --: 2^{nd} \text{ bit flipped } \quad \text{— " — } X_2 \text{ — " —} \\ +-: 3^{rd} \text{ bit flipped } \quad \text{— " — } X_3 \text{ — " —} \end{cases}$