CS 269Q Sect. 2

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Complex #s.

Defn. Real part. Imaginary part.

$$z = x + iy \quad \text{with } i^2 = -1.$$

z is a complex #.

Defn. $z^* = x - iy$

z^* is conjugate of z.

Defn. $|z| = \sqrt{x^2 + y^2}$

$|z|$ is magnitude of z.

Prop.

$$|z| = \sqrt{z^* z}.$$

Exponentials of complex #s.

Note that

$$\frac{d^2}{dx^2} \cos x = -\cos x$$

$$\frac{d^2}{dx^2} \sin x = -\sin x$$

Thus:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Also:

$$\frac{d}{dx} e^x = e^x$$

So:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Now:

$$i^2 = -1$$
$$i^3 = i(i^2) = -i$$
$$i^4 = (i^2)^2 = (-1)^2 = 1.$$

So e^{ix}

$$= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \dots$$

$$= \cos x + i \sin x.$$

Corollary $\quad e^{i\pi} + 1 = 0$

Now take arbitrary

$$z = x + iy$$

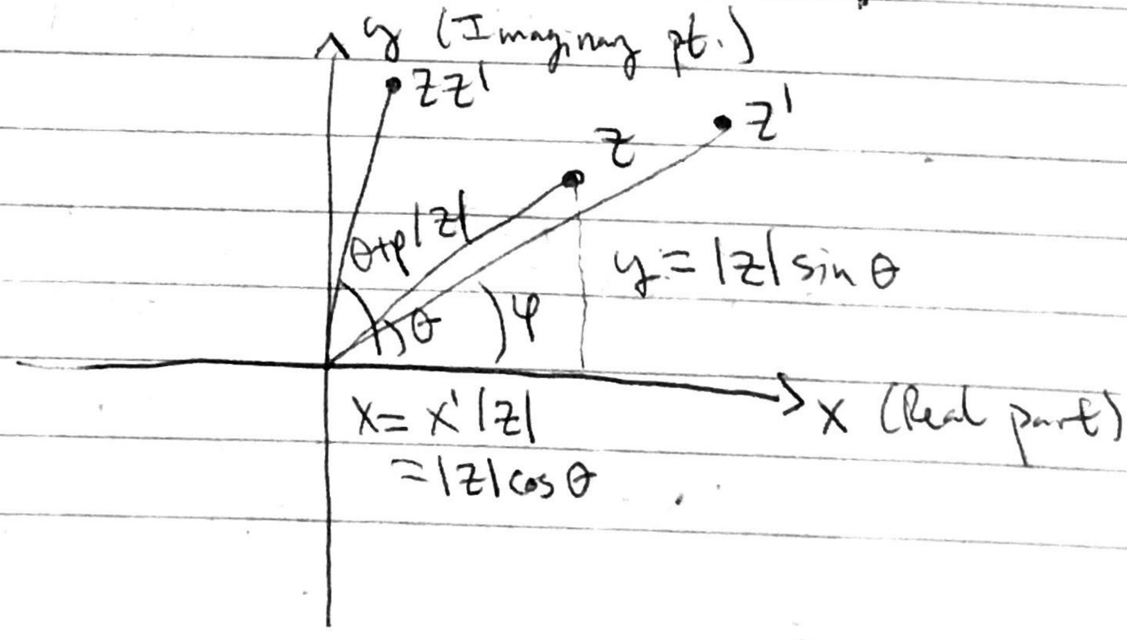
$$= |z| (x' + i y') \qquad x' = x/|z|, \quad y' = y/|z|$$

$$|x' + i y'| = 1.$$

So $x' \geq -1$, $x' \leq 1$.

Thus
$$x' = \cos\theta$$
$$y' = \sin\theta$$

for some $-\pi \leq \theta \lneq \pi$.



In the pic, $z' = |z'| e^{i\varphi}$

So $z z' = |z||z'| e^{i\theta} e^{i\varphi}$

$= |z||z'| e^{i(\theta+\varphi)}$

Magnitudes multiply, angles add.

<u>Claim</u>: overall phase changes ($|\psi\rangle \mapsto e^{i\theta}|\psi\rangle$) make no difference for any observations.

<u>Proof</u>: Measurements m_i, P_i.

$|\psi\rangle \mapsto e^{i\theta}|\psi\rangle$

so $\langle\psi| \mapsto e^{-i\theta}\langle\psi|$.

Thus $p(m_i) = \langle\psi|P_i|\psi\rangle$

$\mapsto \underbrace{e^{-i\theta} e^{i\theta}}_{=e^0 = 1} \langle\psi|P_i|\psi\rangle$

$= \langle\psi|P_i|\psi\rangle$

unchanged.

Bloch sphere

Consider any

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

$|\alpha|^2 + |\beta|^2 = 1$.

So

$$|\psi\rangle = |\alpha| e^{i\gamma} |0\rangle + |\beta| e^{i\delta} |1\rangle.$$

Since $0 \le |\alpha| \le 1$, $\exists\ \theta$ s.t.

$$|\alpha| = \cos \theta/2,$$

with $0 \le \theta \le \pi$.

Since $|\alpha|^2 + |\beta|^2 = 1$,

$$|\beta| = \sin \theta/2.$$

Write

$$|\psi\rangle = e^{i\gamma}\left(\cos \theta/2 \ |0\rangle + e^{i(\delta-\gamma)} \sin \theta/2 \ |1\rangle \right)$$

Delete $e^{i\gamma}$ since makes no diff. by the Claim.

So

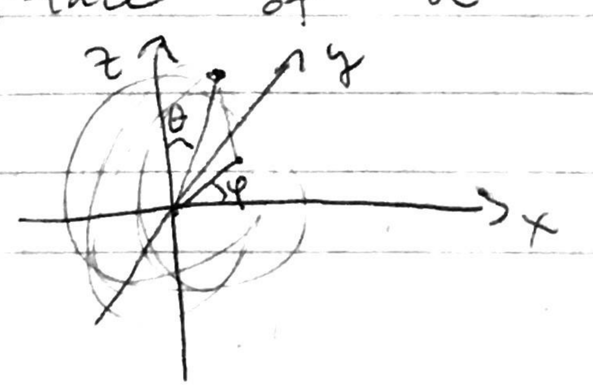
$$|\psi\rangle = \cos\tfrac{\theta}{2} |0\rangle + e^{i\varphi} \sin\tfrac{\theta}{2} |1\rangle$$

where $\varphi = \delta - \gamma$.

Can take $0 \le \varphi < 2\pi$.

So states $|\psi\rangle \longleftrightarrow (\theta, \varphi)$.

Such (θ, φ) trace out the surface of a 3D sphere:



θ = angle from z-axis

φ = angle from x-axis on the projection to pla

The north pole is $\theta = 0$,
$\varphi =$ whatever.

So $|\psi\rangle = \cos 0\, |0\rangle + e^{i\varphi} \sin 0\, |1\rangle$
$= |0\rangle$.

South pole: $\theta = \pi$.

$|\psi\rangle = \cos \pi/2\, |0\rangle + e^{i\varphi} \sin \pi/2\, |1\rangle$
$= |1\rangle$.

<u>Claim:</u> for point (x, y, z) on B.S.,
z is expectation value of Z.

<u>Proof:</u> Note $z = \cos\theta$ from the picture.

Also, $\Pr[\text{measuring } |0\rangle] = |\cos \frac{\theta}{2}|^2$
$= \cos^2 \theta/2$,

$\Pr[\text{measuring } |1\rangle] = |e^{i\varphi} \sin \frac{\theta}{2}|^2$
$= \sin^2 \theta/2$

$$e^{i\theta/2} = \underbrace{\cos \theta/2}_{c} + i \underbrace{\sin \theta/2}_{s}$$

$$(e^{i\theta/2})^2 = e^{i\theta} = \cos\theta + i \sin\theta$$
$$= (c^2 - s^2) + 2ics$$

So $c^2 - s^2 = \cos\theta$

And $c^2 + s^2 = 1$.

Solve by adding/subtracting eqns: $c^2 = \dfrac{1 + \cos\theta}{2}$, $s^2 = \dfrac{1 - \cos\theta}{2}$.

So $\Pr[|0\rangle] = c^2 = \frac{1+\cos\theta}{2} = \frac{1+z}{2}$

$\Pr[|1\rangle] = s^2 = \frac{1-\cos\theta}{2} = \frac{1-z}{2}$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{aligned} E[Z] &= \lambda_1 \Pr[|0\rangle] + \lambda_2 \Pr[|1\rangle] \\ &= \Pr[|0\rangle] - \Pr[|1\rangle] \\ &= 2z/2 = z. \end{aligned}$$

<u>Claim</u>: Same holds for

$E[X] = x, \quad E[Y] = y$

by symmetry.

<u>Claim</u>: Rotation ops

$$R_X(\theta) = \exp(-i\theta X/2)$$
$$R_Y(\theta) = \exp(-i\theta Y/2)$$
$$R_Z(\theta) = \exp(-i\theta Z/2)$$

are rotations abt. x, y, z axes on Bloch sphere resp.

<u>Proof.</u> For any $\sigma = \sigma^\dagger$,

$|\psi\rangle \mapsto \exp(-i\sigma t)|\psi\rangle, \quad t \in \mathbb{R}$

is unitary and conserves $E[\sigma]$:

$$E[\sigma] = \langle\psi|\sigma|\psi\rangle$$

$$\mapsto \langle\psi| \exp(i\underset{=\sigma}{\sigma^\dagger} t)\, \underbrace{\sigma \exp(-i\sigma t)}_{\text{commute}} |\psi\rangle$$

$= \langle \psi | \exp(\underbrace{-i\sigma t + i\sigma t}_{0}) \sigma | \psi \rangle$

$= \langle \psi | \sigma | \psi \rangle$

Rots. about x, y, z conserve $E[X]$, $E[Y]$, $E[Z]$. (prev. claim)

So their ops are

$\exp(-iXt)$, etc.

Note that when $t = \pi$, by Problem 2 of HW 1,

$\exp(-iXt) = -I$.

OTOH, multiplying by -1 has no physical effect, and is therefore a rot. by 2π on Bloch.

Thus for rot. by θ on B.S.,

we have $t = \theta/2$, and get

$R_x(\theta) = \exp(-iX\theta/2)$

as desired. (Left to reader: check handedness of rotations.)

Now we do some exercises. (N & C)

Ex 4.13

Prove: $HXH = Z$

$HZH = X$.

$HYH = -Y$.

Soln First, a lemma about X, Y, Z:

(a). $X^2 = Y^2 = Z^2 = I$.

(b). $XY = iZ$ $\quad$ $YX = -iZ$

$YZ = iX$ $\quad$ $ZY = -iX$

$ZX = iY$ $\quad$ $XZ = -iY$

Also,
$$X = X^\dagger = X^{-1}$$
$$Y = Y^\dagger = Y^{-1}$$
$$Z = Z^\dagger = Z^{-1} \quad \text{(unitary + Hermitian.)}$$

(Prove by direct calculation using the relation $XYZ = iI$.)

Now, $$H = \frac{X+Z}{\sqrt{2}}$$

So $H = H^\dagger$.

Using rels. above,
$$H^2 = \frac{X^2 + Z^2 + \cancel{\overset{-iY}{XZ} + \overset{iY}{ZX}}}{2}$$
$$= (I+I)/2 = I$$

So $H = H^\dagger = H^{-1}$. (unitary + Hermitian)

Thus $$HXH = Z$$
$$\Longleftrightarrow HX = ZH$$
$$\Longleftrightarrow (X+Z)X = Z(X+Z)$$
$$\Longleftrightarrow \underbrace{X^2}_{I} + ZX = ZX + \underbrace{Z^2}_{I}$$
$$\Longleftrightarrow ZX = ZX \quad \checkmark$$

Checking $HZH = X$ is similar

Now $HYH = -Y$

$\Longleftrightarrow HY = -YH$

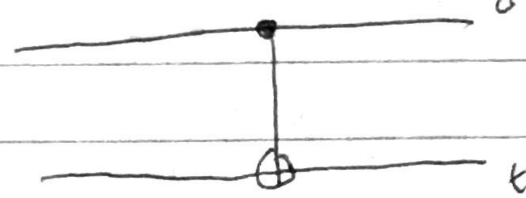
$\Longleftrightarrow (X+Z)Y = -Y(X+Z)$

$\Longleftrightarrow \underset{iZ}{XY} + \underset{-iX}{ZY} = \underset{-(-iZ)}{-YX} \underset{-iX}{-YZ}$

So we're good ✓

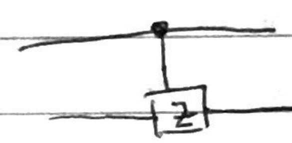
Ex. 4.17

Build

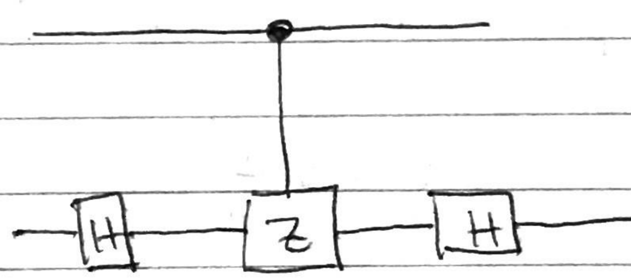


$= c\text{-}X$

from $c\text{-}Z$. $=$



Soln



works. Why?

$|0\rangle|y\rangle \longmapsto |0\rangle|y\rangle = |0\rangle(H^2|y\rangle)$

$|1\rangle|y\rangle \longmapsto |1\rangle(X|y\rangle)$

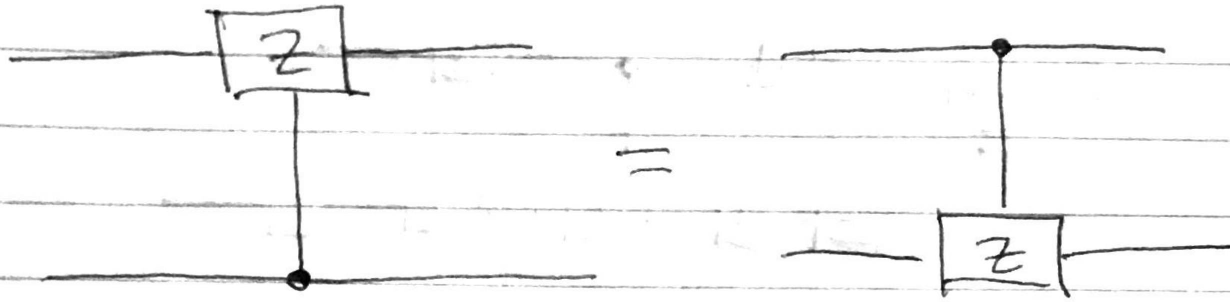
$= |1\rangle(HZH|y\rangle)$ by Ex 4.13

So $|x\rangle|y\rangle \longmapsto$

$|x\rangle(HZ^x H|y\rangle)$

Ex 4.18

Show



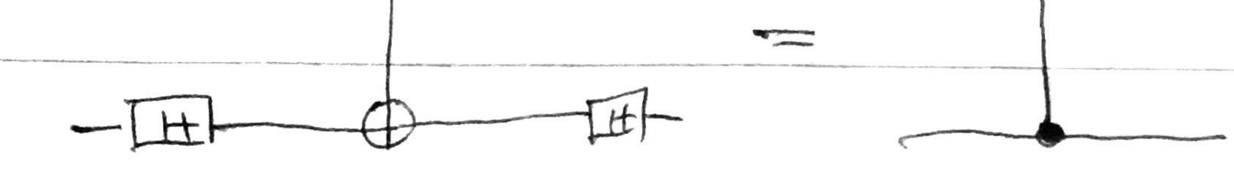
Soln. Check on basis. $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

	LHS picture	RHS picture	
ctrl off	$\lvert 00\rangle \mapsto \lvert 00\rangle$	$\lvert 00\rangle \mapsto \lvert 00\rangle$	ctrl off
control on x (+1)	$\lvert 01\rangle \mapsto \lvert 01\rangle$	$\lvert 01\rangle \mapsto \lvert 01\rangle$	ctrl off
ctrl off	$\lvert 10\rangle \mapsto \lvert 10\rangle$	$\lvert 10\rangle \mapsto \lvert 10\rangle$	ctrl on x(+1)
ctrl on x (−1)	$\lvert 11\rangle \mapsto -\lvert 11\rangle$	$\lvert 11\rangle \mapsto -\lvert 11\rangle$	ctrl on x(−1)

All the same, so ✓

Ex. 4.20

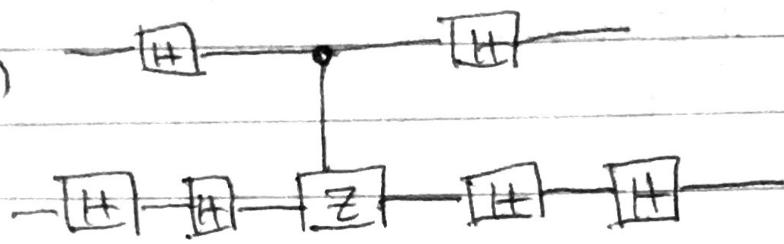
Show



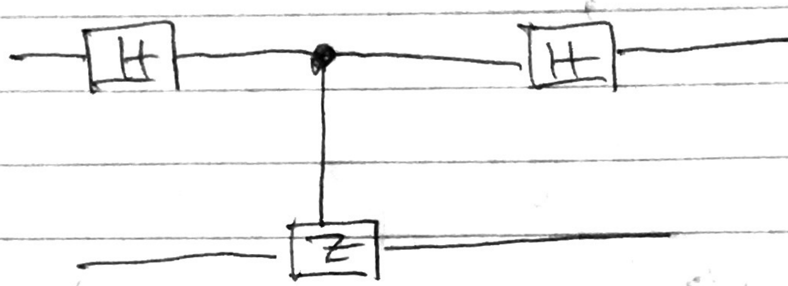
Soln

LHS pic

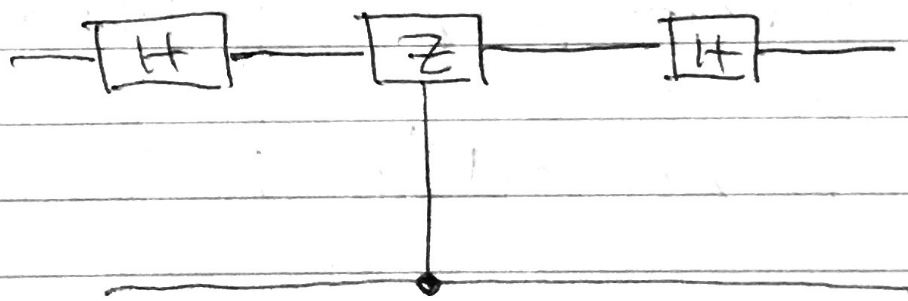
(Ex. 4.17) =



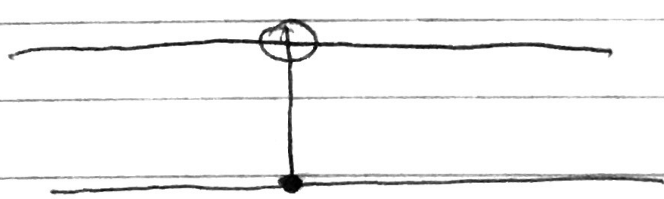
($H^2 = I$) =



(Ex. 4.18) =



(Ex. 4.17, symmetry) =



= RHS pic.

Now define $|+\rangle = H|0\rangle$

$|-\rangle = H|1\rangle$

Note $|0\rangle = H|+\rangle$

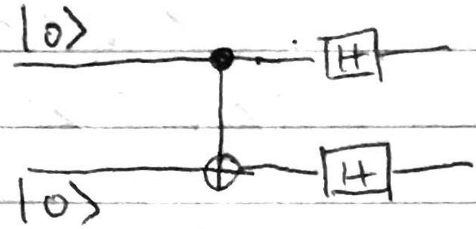
$|1\rangle = H|-\rangle$

1.59

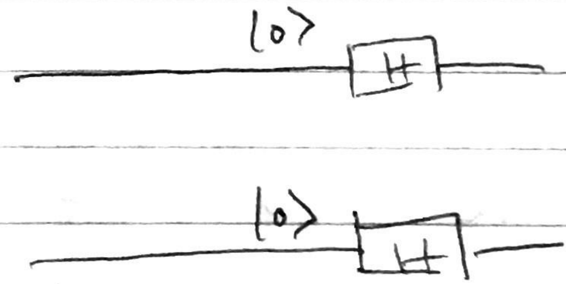
Let us apply the RHS C–NOT gate (1st bit = target, 2nd bit = ctrl) to the $|+\rangle / |-\rangle$ basis.

We have

$|+\rangle|+\rangle$

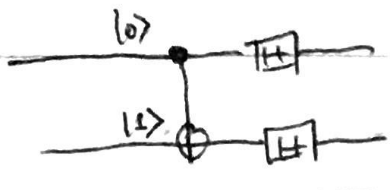
$\longmapsto$ 

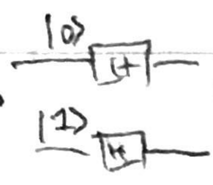
Since $H|+\rangle = |0\rangle$ and LHS pic = RHS pic.

$\longmapsto$ 

$\longmapsto |+\rangle|+\rangle$

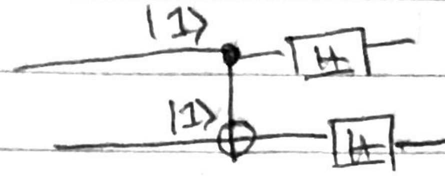
and $|+\rangle|-\rangle$

$\longmapsto$ 

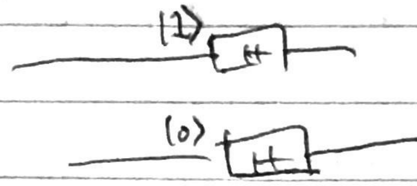
$\longmapsto$ 

$\longmapsto |+\rangle|-\rangle$

Also $|-\rangle|-\rangle \mapsto$



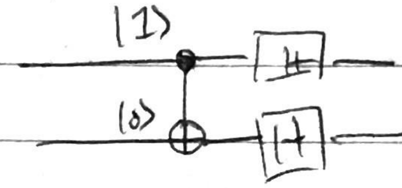
$\mapsto$



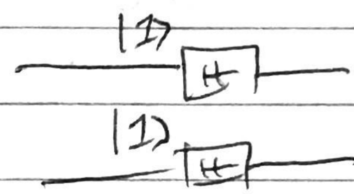
$\mapsto |-\rangle|+\rangle$

and $|-\rangle|+\rangle$

$\mapsto$



$\mapsto$



$\mapsto |-\rangle|-\rangle$

So the 2nd bit flips +/− if 1st bit is −, otherwise doesn't change.

Thus in the +/− basis,

1st bit = ctrl

2nd bit = target,

Flipped from 0/1 basis.

So what you call "ctrl"/"target"

is subjective, i.e. basis-relative. QM itself doesn't care. (All bases are on an equal footing by linearity.)