CS 269Q Section 4

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1 More Bloch sphere

In Section 2, we found that for a point (x, y, z) on the Bloch sphere, we have that $(x, y, z) = (\mathbb{E}(X), \mathbb{E}(Y), \mathbb{E}(Z))$.

It is not too hard to see that given any observable $M = M^\dagger$, the transformation $|\psi\rangle \mapsto \exp(-iMt)\,|\psi\rangle$ gives you a continuous family of unitary transformations that preserve the distribution of M as t sweeps over $\mathbb{R}$. See the Section 2 notes for a proof.

Note that rotations by the x, y, z-axes must conserve their respective coordinate, i.e. $\mathbb{E}(X), \mathbb{E}(Y), \mathbb{E}(Z)$ respectively. Thus we find that $R_x(\theta) = \exp(-iX\theta/2)$ where $R_x(\theta)$ is a counterclockwise rotation by angle θ about the x-axis on the Bloch sphere. For the argument why $t = \theta/2$ and the sign of t, see the Section 2 notes.

What about X, Y, Z themselves? Observe that

$$R_x(\pi) = \exp(-iX\pi/2) = \cos(-\pi/2)I + Xi\sin(-\pi/2) = -iX$$

which is physically identical to X, and thus the same point on the sphere. Here we have used the result from Problem 2 on HW 1, since $X^2 = 1$. Thus X, Y, Z represent rotations by π about their respective axes.

As an exercise in this reasoning, we leave it to the reader to see that H represents a π-rotation about the axis in the $(1, 0, 1)$-direction. Thus, we get geometric interpretations of all the 2×2 gates we have learned so far.

As a final note, consider time-evolution: what happens to a state $|\psi\rangle$ after time t. We have that $|\psi\rangle \mapsto U(t)\,|\psi\rangle$ where $U(t)$ is unitary. Physically, the quantity conserved under time-shifts is the energy, in QM the "Hamiltonian" matrix $\mathcal{H}$. So we find that $U(t) = \exp(-i\mathcal{H}t/\hbar)$, where $\hbar$ is the "reduced Planck's constant," with units of energy times time. It must be included since the argument of a transcendental function has to be unitless, which you can see by expanding the function in a series: if the argument had any unit, then the terms of the series would all have inconsistent units, which is bogus.

2 Experiments to determine T_1, T_2

2.1 Intuition for relaxation times

What do the relaxation times T_1, T_2 correspond to in terms of building a quantum computer? After times on the scale of T_1, we expect a qubit just to end up as $|0\rangle$. After times on the scale of T_2, we expect qubits to lose their relative phases. So T_1 is how much time you have to make a measurement of a state, and T_2 is the time you have to do unitary operations (gates).

The analysis we're going to go into now should make the experimental methods you've been given easier to understand.

2.2 NMR

To explain where the protocols you're given to measure these times come from, we'll consider NMR (nuclear magnetic resonance, Nobel prize in 1944 and 1952) experiments. Incidentally, the medical technique of MRI is the same thing—they actually use the T_1, T_2 times to distinguish between different tissues, bone, etc.

In these experiments, an ensemble of qubits is subjected to a strong static magnetic field, let's say in the z-direction so $\vec{B} = B\hat{z}$. In this situation all their Bloch vectors *precess* about the axis of the magnetic field, i.e. they start to spin around it. We can see the reason for this "Larmor precession" as follows, using the time evolution discussion above. We know that the $|0\rangle$ state is at lower energy than $|1\rangle$. We can also see that $|0\rangle, |1\rangle$ must be eigenkets of the Hamiltonian (energy matrix) $\mathcal{H}_{\text{precess}}$, just from the symmetry of the scenario. Thus $\mathcal{H}$ can be chosen proportional to $-Z$: $-Z$ has $-1, 1$ on the diagonals so this will give you an energy shift between the two states. Then the time evolution is $|\psi\rangle \mapsto \exp(icZt/\hbar)$ for some c. This looks like R_z, so we get a rotation about z-axis as time goes on, i.e. the precession.

Thus we can write the precession as $|\psi\rangle \mapsto R_z(\omega_d t)\,|\psi\rangle$, with ω_d some frequency. This says that after a time t has passed, the state rotates through the angle $\omega_d t$ about the z-axis on the sphere. When $t = 2\pi/\omega_d$ a complete cycle has finished.

The Bloch vectors precess as above, but other things also act upon them (thermal noise, couplings with environment, etc.) We model the effect of all "other things" for a duration $\tau = 50$ ns (the gate time) by simply applying the noisy-I gate. Note that this application is nondeterministic, and we must average results over many simulations (or over our ensemble). So, on average, we have $\exp(-i\mathcal{H}_{\text{other}}\tau/\hbar) = \texttt{NOISY-}I$ where $\mathcal{H}_{\text{other}}$ is the Hamiltonian responsible for all "other things." Raising both sides to t/τ, we see that the overall time evolution here for time t is given by $(\texttt{NOISY-}I)^{t/\tau}$; we just apply the noisy gate for the appropriate number of times.

Thus combining the two effects, after a time t, we have $|\psi\rangle \mapsto \exp(-i(\mathcal{H}_{\text{other}} + \mathcal{H}_{\text{precess}})t/\hbar)\,|\psi\rangle$. Let us suppose additionally that, *on average*, noisy-I commutes with Z. This is justifiable given the symmetry of the experiment about the z-axis. When matrices M_1, M_2 commute you have $\exp(M_1+M_2) = \exp(M_1)\exp(M_2)$.[1] Then we can write the time evolution as just

$$\underbrace{\exp(-i\mathcal{H}_{\text{other}}t/\hbar)}_{(\texttt{NOISY-}I)^{t/\tau}}\underbrace{\exp(-i\mathcal{H}_{\text{precess}}t/\hbar)}_{R_z(\omega_d t)}$$

so the time evolution can be obtained by sequentially applying the noisy gate a number of times, and then letting the precession happen (in either order, since everything commutes).

It turns out that, by applying *pulses*—magnetic fields oscillating at suitable frequencies—we can effect ops like $R_x(\pi)$ and $R_x(\pi/2)$ on the qubits. In our experiments, we interleave pulses and periods of free evolution (precession and noise).

2.3 Measuring T_1

Now our protocol will be as follows:

1. Pulse $R_x(\pi) = X$.
2. Free evolution for time t:
   (a) Noisy-I, applied t/τ times.
   (b) Precess for t: apply $R_z(\omega_d t)$.
3. Measure qubit

Repeating the experiment, we obtain the probability of ending up in $|1\rangle$ (i.e., measuring that Z is -1) at the end. Phenomenologically fit this probability by a decay $A\exp(-t/T_1) + C$. Note that step 2b can have no effect on the final result, since R_z commutes with the measurement Z. Thus we can just remove it, as was done in the earlier spec.

2.4 Measuring T_2

1. Pulse $R_x(\pi/2)$.
2. Free evolution for time t:
   (a) Noisy-I, applied t/τ times.
   (b) Precess for t: apply $R_z(\omega_d t)$.
3. Pulse $R_x(\pi/2)$.
4. Measure qubit

[1] We also used this in showing that the complex exponentials give you unitary transformations that preserve an observable. Exercise: derive a formula for $\exp(M_1)\exp(M_2)$ involving exponential of an infinite series of nested commutators when M_1, M_2 do *not* commute.

What do we expect to see here? Let's set $t = 0$. Then step (1) takes the state to $(0, -1, 0)$ on the sphere, step (2) does nothing, and step 3 takes us to $(0, 0, -1)$. The probability here of getting 1 is 1, so plot the point $(0, 1)$ on the graph. What if $t = \pi/2\omega_d$? Let's also imagine for now that step 2a does nothing, that there is no noise. Step 2b results in a rotation by $\pi/2$ around z, so by now the state is at $(1, 0, 0)$. But the step 3 pulse has no effect on this state. So the probability we get here is 1/2. Continuing this reasoning, at $t = \pi/\omega_d$ we have a probability of 0, at $3\pi/2\omega_d$ a probability of 1/2 again, and $2\pi/\omega_d$ back to the beginning.

Now we add in the noise. After a long time, the noise (2a) tends to take us back to $|0\rangle$. 2b has no effect on this state, and 3 sends it back to the equator of the sphere. Thus after a long time, the probability goes to 1/2.

We can therefore fit the curve by

$$A \exp(-2t/T_2) \sin(\omega_d t - \phi) + C$$

note that we have added in a factor of 2 in front of the time. This is to make T_1, T_2 comparable based on the heuristic reasoning that T_1 should be about twice as long as T_2 (decay requires moving from one pole to the other on the sphere, while the latter is just moving from the equator to a pole).

We can guess what most of these parameters should be, which may help you when curve fitting. We leave it to the reader to use the considerations in the above discussion to find initial guesses for the parameters above.

Because of the different pulses used here (steps 1, 3) compared to the T_1 experiment, note that the precession (step 2b) *cannot* be removed without affecting the resulting curve.

2.5 Choosing ω_d

Physically, ω_d is fixed by the magnetic field strength B_0. You have the freedom to choose it in your assignment. How should we choose a good one? We basically don't want it to be too big or too small. Note that if the last time in any of your experiments is T, then by that time the curve above undergoes $\omega_d T/2\pi$ oscillations, since it takes the sine 2π in its argument for one complete cycle. Choose an ω_d that makes it so that you just have a handful of oscillations between the beginning and the end of the experiment.