Quantum physics: state of an electron \( \psi \in \mathbb{C}^\infty \text{ s.t. } \int |\psi(x)|^2 dx = 1 \). Interpretation: electron is in position \( x \in \mathbb{R} \) with prob \( |\psi(x)|^2 \).

Linear combination of infinitely many classical states \( \Rightarrow \) infinite dim. vector space (Hilbert space).

Quantum computing systems: always finite dim. spaces \( \Rightarrow \) need standard linear algebra.

Ex 1: electron can have top spin (1) or bottom spin (0).

State of system: \( \{\psi_0, \psi_1\} \) such that \( |\psi_0|^2 + |\psi_1|^2 = 1 \).

Represent as vector: \( \psi = (\psi_0, \psi_1) \in \mathbb{C}^2 \), \( \|\psi\| = 1 \).

Ex 2: n binary electrons \( \Rightarrow 2^n \) classical states.

State of system: \( \psi = (\psi_{00}, \psi_{01}, \ldots, \psi_{11}) \in \mathbb{C}^n \), \( \|\psi\| = 1 \).

The realm QP is powerful: amplitudes can be negative (in randomized algs: probs are always positive).
Notation:

1. Column vector \( \psi \in \mathbb{C}^n \): \( |\psi\rangle \) ket
2. Row vector \( \varphi \in \mathbb{C}^n \): \( \langle \varphi | \) bra
3. Inner product between \( |\varphi\rangle \) and \( |\psi\rangle \):
   \[
   \langle \psi | \varphi \rangle = \langle \varphi | \psi \rangle
   \]
4. If \( A \in \mathbb{C}^{m \times n} \) then
   \[
   \begin{bmatrix}
      \text{complex conj of } A \\
      A^T \text{ transpose of } A \\
      A^\dagger \text{ adjoint of } A = (A^T)^* \\
   \end{bmatrix}
   \]
5. If \( A = A^\dagger \) then \( A \) is Hermitian

Thus: if \( A \in \mathbb{C}^{m \times n} \) is Hermitian then \( A \) is diagonalizable.

In particular, let \( U_1, \ldots, U_k \in \mathbb{C}^n \) be eigenspaces of \( A \).

Then \( \mathbb{C}^n = U_1 + \cdots + U_k \) and \( U_i \perp U_j \) for \( i \neq j \).

[A partition \( \mathbb{C}^n \) into orthogonal subspaces]

6. If \( AA^\dagger = I \) then \( A \) is unitary

Thus: if \( U \) is unitary and \( \psi \in \mathbb{C}^n \) then
   \[
   \| \psi \|^2 = \| A \psi \|^2
   \]

Examples:

- Single q-bit, state = \( |\psi\rangle \in \mathbb{C}^2 \)
- Basis of \( \mathbb{C}^2 \): \( |0\rangle \triangleq (1,0) \in \mathbb{C}^2 \)
  \( |1\rangle \triangleq (0,1) \in \mathbb{C}^2 \)

\[
|\psi\rangle = \psi_0 \cdot |0\rangle + \psi_1 \cdot |1\rangle,
\]
\[
|\psi\rangle | \psi\rangle^\dagger = \frac{1}{|\langle \psi | \psi \rangle|}
\]

n-qubits: basis \( |00\ldots0\rangle, |00\ldots1\rangle, \ldots, |11\ldots\rangle \) (\( 2^n \) vectors)
**Pauli matrices:**
\[
X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

\[
X \cdot |\psi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_0 \end{bmatrix} = |\psi_1\rangle \otimes |\psi_0\rangle
\]

(Not gate)

\[
Z \cdot |\psi\rangle = |\psi_1\rangle \otimes |\psi_0\rangle - |\psi_0\rangle \otimes |\psi_1\rangle
\]

Note: \(X, Y, Z\) are Hermitian & Unitary

**More notation:**

1. \(\langle \psi | A | \psi \rangle = \langle \psi | A \psi \rangle = \langle A^\dagger \psi | \psi \rangle\)
2. **Outer product:** \(|\psi\rangle \otimes |\psi_j\rangle \in \mathbb{C}^{n \times n}\)

\[
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle
\]

**Tensor product:**
\[
|\psi\rangle = |\psi_0\rangle \otimes |\psi_1\rangle
\]

\[
|\phi\rangle = |\phi_0\rangle \otimes |\phi_1\rangle
\]

**Description of composite system:**

State basis: \(|00\rangle, |01\rangle, |10\rangle, |11\rangle \in \mathbb{C}^4\)

\[
|\psi\rangle \otimes |\phi\rangle \overset{def}{=} |\psi\rangle \otimes |\phi\rangle = |\psi_0\phi_0\rangle \otimes |00\rangle + |\psi_0\phi_1\rangle \otimes |01\rangle + |\psi_1\phi_0\rangle \otimes |10\rangle + |\psi_1\phi_1\rangle \otimes |11\rangle
\]

In general: \(|\psi\rangle \in n\text{-dim}, \quad |\phi\rangle \in m\text{-dim} \Rightarrow |\psi\rangle \otimes |\phi\rangle \in n \times m\text{-dim}\)
Postulates of quantum mechanics

(1) An isolated physical system is fully described by a unit state vector $|\psi\rangle$ in a state space $\mathbb{C}^n$.

(2) If $|\psi_1\rangle$ describes an isolated system at time $t_1$ and $|\psi_2\rangle$ at time $t_2$, then $|\psi_2\rangle = U \cdot |\psi_1\rangle$ where $U$ is unitary.

(3) Measurement: projection into a linear subspace.

Let $M \in \mathbb{C}^{n \times n}$ be Hermitian matrix (observable) and let $U_1, \ldots, U_k \in \mathbb{C}^n$ be its orthogonal eigenspaces ($\mathbb{C}^n = U_1 \oplus \cdots \oplus U_k$) and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ be its distinct eigenvalues.

Let $P_i \in \mathbb{C}^{n \times n}$ be a projection matrix onto $U_i$, where $P_i + \cdots + P_k = I$ (so $P_i |\psi\rangle + \cdots + P_k |\psi\rangle = |\psi\rangle$).

Then measuring $|\psi\rangle$ using $M$ gives outcome $\lambda_i \in \mathbb{C}$ with prob.

$$q_i \overset{\text{def}}{=} \frac{|P_i \cdot |\psi\rangle|^2}{\text{norm of projection squared}} \quad \left( \sum_i q_i = 1 \right)$$

If the outcome is $\lambda_i$, then the system is in state

$$\left( \frac{P_i \cdot |\psi\rangle}{\sqrt{q_i}} \right).$$

Fact: $E[M] = \langle \psi | M | \psi \rangle$ normalized projection (norm 1)
Example: EPR pair \( \psi = \frac{\ket{00} + \ket{11}}{\sqrt{2}} \)

Measure first bit:

- \( U_0 = \text{span}(\ket{00}, \ket{10}) \subseteq \mathcal{C}_4 \)
- \( U_1 = \text{span}(\ket{11}, \ket{11}) \subseteq \mathcal{C}_4 \)

As a Hermitian matrix, take \( n = 2 \) (it operating on first bit)

\[ \psi = \frac{\ket{00} + \ket{11}}{\sqrt{2}} \implies \Pr(\chi_0) = \Pr(\chi_1) = \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2} \]

\( \chi_0 = 1 \)

\( \chi_1 = -1 \)

\( 0 \quad 1 \)

\( 1_0 \quad 1_1 \)

If measured 0, state will be \( \psi' = \ket{10} \)

If measured 1, state will be \( \psi' = \ket{11} \)

\( \implies \) spooky action at a distance.

Superdense coding: sending two bits for the price of one qubit

A & B have an EPR pair \( \frac{\ket{00} + \ket{11}}{\sqrt{2}} \)

A has first qubit, B has second qubit.

A wants to send two bits to Bob:

1. She operates on her own qubit: 00 \( \xleftrightarrow{X} \) 01, 10 \( \xleftrightarrow{Y} \) 11
2. Send qubit to Bob (one qubit)
3. Bob obtains: 00 \( \frac{\ket{00} + \ket{11}}{\sqrt{2}} \), 01 \( \frac{\ket{10} - \ket{11}}{\sqrt{2}} \), 10 \( \frac{\ket{10} + \ket{11}}{\sqrt{2}} \), 11 : \( \frac{\ket{10} - \ket{11}}{\sqrt{2}} \)

\( \implies \) This is an orthonormal basis, so Bob can measure and recover both bits.