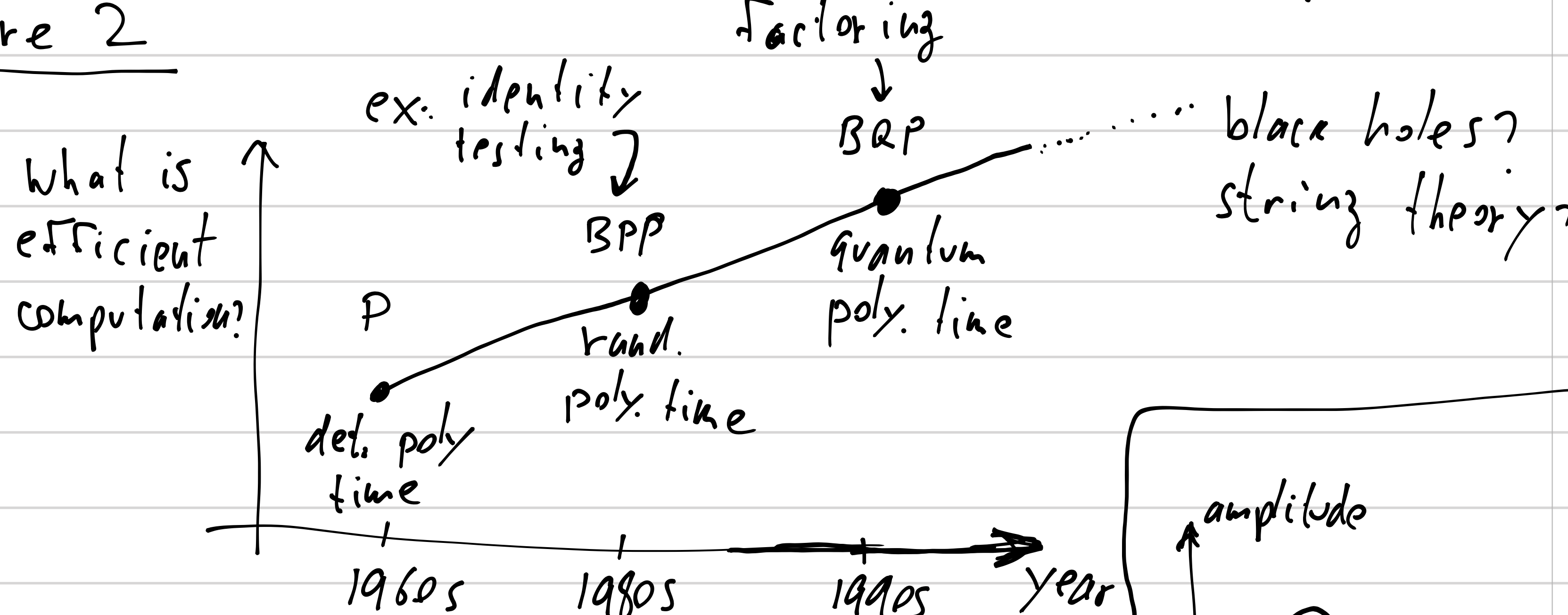
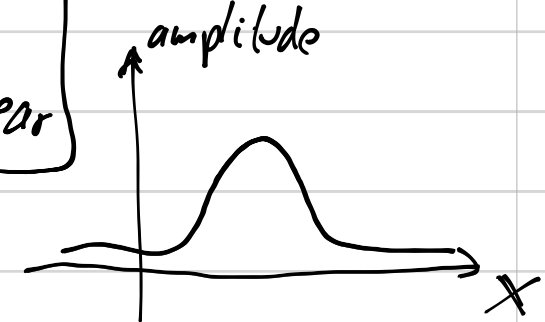
Lecture 2

Apr. 2019





Quantum physics: state of an electron

$\psi : \mathbb{R} \to \mathbb{C}$ s.t. $\int |\psi(x)|^2 dx = 1$

Interpretation: electron is in position $x \in \mathbb{R}$ with prob. $|\psi(x)|^2$.

linear combination of infinitely many classical states

$\Rightarrow$ infinite dim. vector space (Hilbert space)

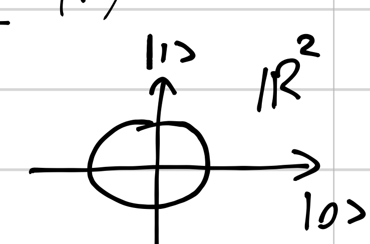
Quantum computing systems: always finite dim. spaces

$\Rightarrow$ need standard linear algebra.

ex 1: electron can have top spin (1) or bottom spin (0)

state of system: $\begin{cases} \psi_0 & \text{amplitude in state } 0 \\ \psi_1 & \text{— " — in state } 1 \end{cases}$

s.t. $|\psi_0|^2 + |\psi_1|^2 = 1$



Represent as vector: $\psi = (\psi_0, \psi_1) \in \mathbb{C}^2$, $\|\psi\|_2 = 1$.

ex 2: n binary electrons $\Rightarrow 2^n$ classical states

state of system: $\psi = (\psi_{00 0}, \psi_{001}, \dots, \psi_{11 1}) \in \mathbb{C}^{(2^n)}$, $\|\psi\|_2 = 1$.

The reason QP is powerful: amplitudes can be negative.

(in randomized algs: probs are always positive)

Read ch. 2.1, 2.2, 2.3, 2.6 !

Notation:

(1) column vector $\psi \in \mathbb{C}^n$: $|\psi\rangle$ ket

row vector $\varphi \in \mathbb{C}^n$: $\langle \varphi^* |$ bra

(2) $\langle \varphi | \psi \rangle$ inner product between $|\varphi\rangle$ and $|\psi\rangle$.

$$[\ \varphi\ ] \cdot \begin{bmatrix} \psi \end{bmatrix} = \langle \varphi | \psi \rangle = \sum_{i=1}^{n} \varphi_i^* \cdot \psi_i$$

(3) if $A \in \mathbb{C}^{n \times n}$ then $\begin{cases} A^* & \text{complex conj of } A \\ A^T & \text{transpose of } A \\ A^\dagger & \text{adjoint of } A = (A^T)^* \end{cases}$

(4) if $A = A^\dagger$ then A is <u>Hermitian</u>

<u>Thm</u>: if $A \in \mathbb{C}^{n \times n}$ is Hermitian then A is <u>diagonalizable</u>

In particular, let $U_1, \dots, U_k \subseteq \mathbb{C}^n$ be eigenspaces of A.

Then $\mathbb{C}^n = U_1 + \dots + U_k$ and $U_i \perp U_j$ for $i \neq j$.

[A partions $\mathbb{C}^n$ into orthogonal subspaces]

(5) if $A \cdot A^\dagger = I$ then A is <u>unitary</u>

<u>Thm</u>: if A is unitary and $\psi \in \mathbb{C}^n$ then

$$\|\psi\|_2 = \|A \cdot \psi\|_2$$

<u>examples</u>: single q-bit, state $= |\psi\rangle \in \mathbb{C}^2$

basis of $\mathbb{C}^2$: $|0\rangle \overset{\text{def}}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$

$|1\rangle \overset{\text{def}}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$

$|\psi\rangle = \psi_0 \cdot |0\rangle + \psi_1 \cdot |1\rangle$, $|\psi_0|^2 + |\psi_1|^2 = \langle \psi | \psi \rangle = 1$

n-qbits: basis $|00\dots0\rangle, |00\dots1\rangle, \dots, |11\dots1\rangle$ (2^n vectors)

<u>Pauli matrices</u>: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$X \cdot |\psi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = \psi_1 \cdot |0\rangle + \psi_0 \cdot |1\rangle$$

(not gate)

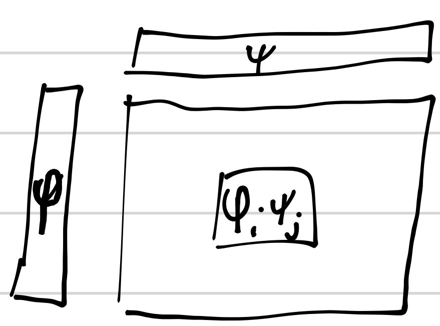
$$Z \cdot |\psi\rangle = \psi_1 \cdot |0\rangle - \psi_1 \cdot |1\rangle$$

Note: X, Y, Z are Hermitian & Unitary.

<u>more notation</u>:

(1) $\langle \varphi | A | \psi \rangle = \langle \varphi | A \psi \rangle = \langle A^{\dagger} \varphi | \psi \rangle$

(2) outer product: $|\varphi\rangle\langle\psi| \in (\varphi_i \psi_j) \in \mathbb{C}^{n \times n}$



ex: $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \overbrace{|0\rangle \cdot \langle 0|}^{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} - \overbrace{|1\rangle\langle 1|}^{\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}}$

<u>Tensor product</u>: $|\psi\rangle = \psi_0 \cdot |0\rangle + \psi_1 \cdot |1\rangle$ ← one qbit

$|\varphi\rangle = \varphi_0 \cdot |0\rangle + \varphi_1 \cdot |1\rangle$ ← another qbit

description of composite system:

basis = $|00\rangle, |01\rangle, |10\rangle, |11\rangle \in \mathbb{C}^4$

state

$$|\psi\rangle|\varphi\rangle \overset{\text{def}}{=} |\psi\rangle \otimes |\varphi\rangle = \psi_0\varphi_0 |00\rangle + \psi_0\varphi_1 |01\rangle + \psi_1\varphi_0 |10\rangle + \psi_1\varphi_1 |11\rangle$$

In general: $|\psi\rangle$ n-dim, $|\varphi\rangle$ m-dim $\Rightarrow$ $|\psi\rangle|\varphi\rangle$ - $n \cdot m$ dim

Postulates of quantum mechanics

(1) An isolated physical system is fully described by a unit state vector $|\psi\rangle$ in a state space $\mathbb{C}^n$.

(2) if $|\psi_1\rangle$ describes $\underbrace{\text{an system}}^{\text{isolated}}$ at time t_1 and $|\psi_2\rangle$ at time t_2 then $|\psi_2\rangle = U \cdot |\psi_1\rangle$ where U is <u>unitary</u> that does depend on $|\psi_1\rangle, |\psi_2\rangle$.
(U depends only on the system)

(3) <u>Measurement</u>: projection into a linear subspace.

Let $M \in \mathbb{C}^{n \times n}$ be Hermitian matrix. (observable)

and let $U_1, \ldots, U_K \subseteq \mathbb{C}^n$ be its $\overbrace{\text{eigenspaces}}^{\text{orthogonal}}$ $(\mathbb{C}^n = U_1 \oplus \cdots \oplus U_K)$
$\lambda_1, \ldots, \lambda_K \in \mathbb{C}$ be its distinct eigenvalues.

Let $P_i \in \mathbb{C}^{n \times n}$ be a projection matrix into U_i,

where $P_1 + \cdots + P_K = I$ $\left(\text{so } P_1|\psi\rangle + \ldots + P_K|\psi\rangle = |\psi\rangle\right)$.

Then measuring $|\psi\rangle$ using M gives outcome $\lambda_i \in \mathbb{C}$

with prob. $q_i \overset{\text{def}}{=} \underbrace{\| P_i \cdot |\psi\rangle \|^2}_{\text{norm of projection squared}}$ $(q_1 + \ldots + q_K = 1)$

If the outcome is λ_i then the system

is in state $\underbrace{\left(\frac{P_i \cdot |\psi\rangle}{\sqrt{q_i}}\right)}_{\text{normalized projection (norm 1)}}$.

Fact: $E[M] = \langle \psi | M | \psi \rangle$

example: EPR pair $\psi = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

measure first bit : $U_0 = \text{span}(|00\rangle, |01\rangle) \subseteq \mathbb{C}^4$
$U_1 = \text{span}(|10\rangle, |11\rangle) \subseteq \mathbb{C}^4$

as a Hermitian matrix, take $M = Z_1$ (Z operating on first bit)

$$\psi = \underset{\substack{\Uparrow \\ U_0}}{\frac{|00\rangle}{\sqrt{2}}} + \underset{\substack{\Uparrow \\ U_1}}{\frac{|11\rangle}{\sqrt{2}}} \Rightarrow \Pr[\underset{\text{"0"}}{\lambda_0}] = \Pr[\underset{\text{"1"}}{\lambda_1}] = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \qquad \begin{array}{l}\lambda_0 = 1 \\ \lambda_1 = -1\end{array}$$

if measured 0, state will be $\psi' = |00\rangle$

if measured 1, state will be $\psi' = |11\rangle$

$\Rightarrow$ spooky action at a distance.

Superdense coding: sending two bits for the price of one qbit

A & B have an EPR pair $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

A has first qbit. B has second qbit.

A wants to send two bits to Bob:

(1) She operates on her own qbit: 00: I 01: Z ← flip 1
10: X ← not   11: iY

(2) Send qbit to Bob (one qbit)

(3) Bob obtains: 00: $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$; 01: $\frac{|00\rangle - |11\rangle}{\sqrt{2}}$; 10: $\frac{|10\rangle + |01\rangle}{\sqrt{2}}$

11: $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$ $\Rightarrow$ This is an orthonormal basis, so Bob can measure and recover both bits.